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THE METHOD OF TWO-SIDED TRANSFER OF PARAMETERS FOR SOLVING BOUNDARY VALUE PROBLEMS OF THE THEORY OF OSCILLATIONS

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Shortcoming of the method of initial parameters when used for determining oscillations of branched systems including those in which the boundary effect is present are pointed out. It is shown that these shortcomings are eliminated by subjecting the selection of parameters that define the state of a given section of the system, and of equations defining each parameter to specific conditions that are natural for the considered system. It is, then, as a rule to aaply the two-sided carry-over of parameters. It is further shown that the calculation of oscillations of variable cross section bars is considerably simplified by using the method of parameter carry-over in conjuction with the method of slow varying coeffcicents. Transverse oscillations of variable cross section bars are considered as an example.

1. The investigation of stable oscillation of elastic bar systems and of a number of other phenomena reduces to the totality of problems of the determination of vector functions each of which depends on a single coordinate and is defined either by a system of ordinary linear differetnial equations (along sections where parameters of an elastic system are continuous) or by conditions of conjustion (at the boundaries of continuity sections). These relationships are supplemented by conditions specified at the boundaries of a given function. A widely used device of solving such boundary value problems consists of reducing these to Cauchy problems using the method of initial parameters [1,4]. The latter method is, however, specifically adapted to chain systems in which the number of degrees of freedom of sections do not vary with the longitudinal coordinate. In the calculation of system with branches and intermediate supports the number of conditions at the beginning of a section which contain its parameters may not coincide with the number of such paramaters. Hence it becomes necessary to resort to devices alien to the idea of initial parameter method [2—6].

Difficulties of another kind arise in systems with explicit boundary effect (boundary layer) [4,7,8], when the solution contains exponentially increasing and decreasing functions. If for each of these $\lambda_1 l_1 + \ldots + \lambda_n l_n \leqslant 1$ (n is the number of sections of the system, λ_j is the absolute value of the exponent of the exponential function along the j-th section, and l_j the length of the latter), the reduction of the boundary value problem to that of Cauchy unavoidably leads to poorly defined systems [4-6]. It is, thus, not possible to use in systems where the boundary effect is present the methods of one-sided parameter carry-over [4] for which all parameters of a given section of the system are determined by conditions established at that section beginning.

It follows from the above that the selection of parameters which define the state of a system and of equations that define each parameter must be subjected to specific

conditions natural for that system. For instance, for systems in which the boundary effect is present these parameters are to be selected so as to prevent mixing of boundary effects related to different boundaries of a particular section of the system. Thus, the parameter that defines the boundary layer at a particular cross section must not be based on conditions derived for some other corss section. The idea of such approach is close to that which is at the base of the asymptotic method proposed by Bolotin [7 — 9] for calculating oscillations of bars, plates and shells.

2. As an illustration of the proposed method, which can be described as the method of two-sided carry-over of parameters (cf. [4]), we present the sequence of calculation of free oscillations of a beam consisting of three constant cross sections. We assume that a concentrated load whose mass and moment of inertia are m_1^* and J_1, is applied between the first and second sections and an intermediate support is located between the second and third. We specify solutions for each section in the form [1]

$$Y_j(x_j) = \Psi_j(x_j)A_j$$
$$Y_j = \{v_j, L_j, M_j, N_j\}, \quad A_j = \{a_j, b_j, p_j, q_j\} \tag{2.1}$$

$$\Psi_j = \begin{Vmatrix} \sin\Lambda & \cos\Lambda & e^{-\Lambda_x} & e^{\Lambda_x-\Lambda_l} \\ \lambda\cos\Lambda & -\lambda\sin\Lambda & -\lambda e^{-\Lambda_x} & \lambda e^{\Lambda_x-\Lambda_l} \\ -c\lambda^2\sin\Lambda & -c\lambda^2\cos\Lambda & c\lambda^2 e^{-\Lambda_x} & c\lambda^2 e^{\Lambda_x-\Lambda_l} \\ -c\lambda^3\cos\Lambda & c\lambda^3\sin\Lambda & -c\lambda^3 e^{-\Lambda_x} & c\lambda^3 e^{\Lambda_x-\Lambda_l} \end{Vmatrix}_j \tag{2.2}$$

$$\Lambda_j = \Lambda_j(x_j) = \lambda_j x_j + \varphi_j, \quad \Lambda_{jx} = \lambda_j x_j, \quad \Lambda_{jl} = \lambda_j l_j, \quad \lambda_j = \left(\frac{\omega^2 m_j}{c_j}\right)^{1/4}$$

where Y_j is vector (column matrix) of state (v_j is the deflection, $L_j = v_j'$ is the angle of turn, $M_j = c_j L_j'$ is the bending moment, $N_j = M_j'$ is the shear force), c_j is the flexural rigidity, m_j is the running mass, Ψ_j is the matrix of solutions, and A_j is the vector of constants of integration.

Using the two equations that follow from boundary conditions at end $x_1 = 0$ we express coefficients b_1 and p_1 (function $p_1 \exp(-\lambda_1 x_1)$ defines the boundary layer at cross section $x_1 = 0$) in terms of a_1 and q_1

$$b_1 = \mu_{21}a_1 + \nu_{21}E_1 q_1, \quad p_1 = \mu_{31}a_1 + \nu_{31}E_1 q_1, \quad E_j = \exp(-\lambda_j l_j) \tag{2.3}$$

Phase φ_1 is selected so that the inequality $|\mu_{21}| < 1$ is satisfied. For instance, when a hinged support is located at cross section $x_1 = 0$, then assuming $\varphi_1 = 0$, we obtain $\mu_{21} = \nu_{21} = \nu_{31} = 0$, and $\nu_{31} = -1$.

The conditions at junctions of adjacent sections of a bar are

$$v_1(l_1) - v_2(0) = 0, \quad L_1(l_1) - L_2(0) = 0 \tag{2.4}$$
$$M_1(l_1) - J_1\omega^2 L_1(l_1) - M_2(0) = 0, \quad N_1(l_1) + m_1^*\omega^2 v_1(l_1) - N_2(0) = 0$$
$$v_2(l_2) = 0, \quad v_3(0) = 0, \quad L_2(l_2) - L_3(0) = 0, \quad M_2(l_2) - M_3(0) = 0 \tag{2.5}$$

Taking into consideration that the boundary effect at cross section $x_2 = 0$ determines functions which contain q_1 and p_2, from conditions (2.4) we obtain ($\varphi_2 = 0$)

$$\psi_{11}^q q_1 - b_2 - p_2 = -\psi_{11}^a a_1 + E_2 q_2, \quad \psi_{13}^q q_1 - a_2 + p_2 = \tag{2.6}$$
$$-\psi_{12}^a a_1 + E_2 q_2, \quad \psi_{13}^q q_1 + b_2 - p_2 = -\psi_{13}^a a_1 + E_2 q_2$$
$$\psi_{14}^q q_1 + a_2 + p_2 = -\psi_{14}^a a_1 + E_2 q_2$$
$$(\psi_{11}^a = S_1 + \mu_{21} C_1 + \mu_{31} E_1, \quad \psi_{11}^q = 1 + \nu_{21} E_1 C_1 + \nu_{31} E_1^2$$
$$\psi_{12}^a = \lambda_1 \lambda_2^{-1} (C_1 - \mu_{21} S_1 - \mu_{31} E_1), \ldots, S_j = \sin \Lambda_j(l_j), \quad C_j = \cos \Lambda_j(l_j))$$

from which follows

$$q_1 = \mu_{41} a_1 + \nu_{41} E_2 q_2, \quad a_2 = \mu_{12} a_1 + \nu_{12} E_2 q_2 \tag{2.7}$$
$$b_2 = \mu_{22} a_1 + \nu_{22} E_2 q_2, \quad p_2 = \mu_{32} a_1 + \nu_{32} E_2 q_2$$

where the coefficients $\mu_{41}, \ldots, \mu_{32}$ and $\nu_{41}, \ldots, \nu_{32}$ are solutions of system (2.6) by substituting $-\psi_{11}^a, \ldots, -\psi_{14}^a$ and $1, \ldots, 1$. respectively, into its right-hand side.

The boundary effect at cross section $x_3 = 0$ is defined by functions that contain q_2 and p_3. Hence conditions (2.5) with allowance for relationships (2.7) yield

$$\psi_{21}^q q_2 = -\psi_{21}^a a_1, \quad b_3 + p_3 = -E_3 q_3 \tag{2.8}$$
$$\psi_{22}^q q_2 - a_3 + p_3 = -\psi_{22}^a a_1 + E_3 q_3$$
$$\psi_{23}^q q_2 + b_3 - p_3 = -\psi_{23}^a a_1 + E_3 q_3$$
$$(\psi_{21}^a = \mu_{12} S_2 + \mu_{22} C_2 + \mu_{32} E_2$$
$$\psi_{21}^q = 1 - \nu_{12} E_2 S_2 + \nu_{22} E_2 C_2 + \nu_{32} E_2^2$$
$$\psi_{22}^a = \lambda_2 \lambda_3^{-1} (\mu_{12} C_2 - \mu_{22} S_2 - \mu_{32} E_2), \ldots)$$

when $\varphi_3 = 0$. From Eqs. (2.8) we have

$$q_2 = \mu_{42} a_1 + \nu_{42} E_3 q_3, \quad a_3 = \mu_{13} a_1 + \nu_{13} E_3 q_3 \tag{2.9}$$
$$b_3 = \mu_{23} a_1 + \nu_{23} E_3 q_3, \quad p_3 = \mu_{33} a_1 + \nu_{33} E_3 q_3$$

where $\mu_{42}, \ldots, \mu_{33}$ and $\nu_{42}, \ldots, \nu_{33}$ are solutions of system (2.8) when ψ_{21}^a, $0, \psi_{22}^a, \psi_{23}^a$ and $0, -1, 1, 1.$ are, respectively, substituted into its right-hand side.

Taking into account formulas (2.9), from the boundary conditions at the end $x_3 = l_3$ we obtain two equations of the form

$$\psi_{31}^a a_1 + \psi_{31}^q q_3 = 0, \quad \psi_{32}^a a_1 + \psi_{32}^q q_3 = 0 \tag{2.10}$$

The frequencies ω for which the determinant of that system vanishes corresponds to one of the principal oscillations of the system. The form of these oscillations can be determined from formulas (2.10), (2.9), (2.7), and (2.3).

The increase of the order number of principal oscillations does not lead in the described method to any particular computation difficulties, since the order of the coefficients μ_{rj} and ν_{rj} remains unchanged with increasing frequency. Moreover in the case of fairly high frequencies with the numbers $E_j \ll 1$, it is possible to disregard the terms that contain these, and this considerably simplifies calculations (cf. [9]). Note also that unlike in the method of initial parameters in the proposed method the presence of an intermediate support only simplifies the calculation.

In the calculation of low-frequency oscillations when the appearance of poorly

defined systems does not present misgivings, there is no need to separate the boundary effect. In that case simpler equations are obtained by using fundamental systems of solutions. In the case of transverse oscillations of bars such system is defined by Krylov's functions [1].

Let us revert to the three-span beam considered above. Along each of its section we have

$$v_j = a_{j1}S(\lambda_j x_j) + a_{j2}T(\lambda_j x_j) + a_{j3}U(\lambda_j x_j) + a_{j4}V(\lambda_j x_j)$$

Assuming that the left-hand end of the beam is hinged and using the conditions at the junction of the first and second bars, we obtain linear relationships of the form

$$a_{2r} = \sigma_{2r}a_{12} + \rho_{2r}a_{14} \quad (r = 1, 2, 3, 4)$$

We express the equations which follow from conditions (2.3) as

$$\psi_{11}^{(4)}a_{14} = -\psi_{11}^{(2)}a_{12}, \quad a_{21} = 0$$

$$\psi_{12}^{(4)}a_{14} - a_{32} = -\psi_{12}^{(2)}a_{12}, \quad \psi_{13}^{(4)}a_{14} - a_{33} = -\psi_{13}^{(2)}a_{12}$$

$$(\psi_{11}^{(2)} = \sigma_{21}S_2 + \ldots + \sigma_{24}V_2, \ \psi_{11}^{(4)} = \rho_{21}S_2 + \ldots + \rho_{24}V_2$$

$$\psi_{12}^{(2)} = \lambda_2\lambda_3^{-1}(\sigma_{21}V_2 + \sigma_{22}S_2 + \sigma_{23}T_2 + \sigma_{24}U_2), \ldots)$$

Parameter a_{34} remains arbitrary. The state of the third section is, thus, defined by the two parameters a_{12} and a_{34}. (Similar results are obtained by the method of intermediate supports in the method of initial parameters in [3]. It requires, however, extensive supplementary explanations).

Boundary conditions at the end $x_3 = l_3$ yield the system

$$\psi_{31}^{(2)}a_{12} + \psi_{31}^{(4)}a_{34} = 0, \quad \psi_{32}^{(2)}a_{12} + \psi_{32}^{(4)}a_{34} = 0$$

which makes possible the determination of the frequency and form of principal oscillations.

3. In calculating branched systems it is necessary, first of all, to consider the branches between nodes. One of such branches of a bar system is shown by AB in Fig. 1. It is assumed that perturbing effects are applied only at the segment boundaries.

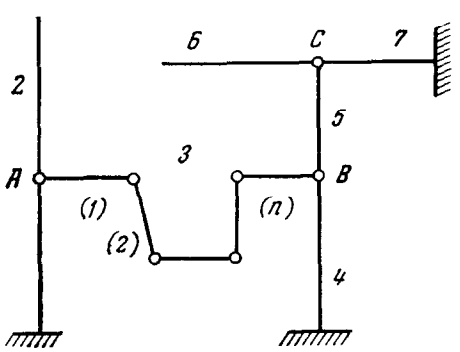

Fig. 1

Let the vector of state $Y_j(x_j)$ of each of the n sections of branch AB be defined by formula (2.1) which contains the vector of constant parameters $A_j = \{a_{j1}, \ldots, a_{jm}\}$. When the boundary effect is absent or weak, then, using the conditions at section joints, it is necessary to obtain the relationships

$$A_j = K_jA_1 + A_j^F \quad (j = 2, \ldots, n) \qquad (3.1)$$

or ($A_j = K_j^*A_n + A_j^F$, $j = 1, \ldots, n-1$), where $K_j^{(*)}$ is the square

matrix of the carry-over and vector A_j^F appears in the presence of perturbation.

Let us assume for definiteness that in the presence of the boundary effect (that vector) is defined at the left- and right-hand ends of the j-th section by parameters $a_{j,\,m-1}$ and a_{jm}. Using the conditions at the segment junction it is then necessary to determine the relationships

$$A_j = K_j' \{A_1^{m-1},\, a_{nm}\} + A_j^F \quad (j = 1,\, 2,\, \ldots,\, n) \tag{3.2}$$

Here and in what follows $A_j^{m-1} = \{a_{j1},\, \ldots,\, a_{j,m-1}\}$. The square matrices K_j' can be constructed as follows. Let the system of equations defining the conditions at the junction of the j-th and the $j+1$-st sections be of the form $\Psi_j^l A_j + \Psi_{j+1}^\circ A_{j+1} = F_j$ or more precisely

$$\psi_{j1}^l a_{j1} + \ldots + \psi_{jm}^l a_{jm} + \psi_{j+1,\,1} a_{j+1,\,1} + \ldots + \overset{\circ}{\psi}_{j+1,\,m} a_{j+1,\,m} = F_j$$

where $\psi_{jr}^{\ l}$ and $\overset{\circ}{\psi}_{j+1,r}$ are columns of matrices Ψ_j^l and Ψ_j° (the latter generally differ from $\Psi_j(l_j)$ and $\Psi_j(0)$) and vector F_j defines the perturbation effect. We write the obtained system as follows:

$$\Psi_{j+1}^{0,\,m-1} A_{j+1}^{m-1} + \psi_{jm}^l a_{jm} = -\Psi_j^{l,\,m-1} A_j^{m-1} - \overset{\circ}{\psi}_{j+1,\,m} a_{j+1,\,m} + F_j \tag{3.3}$$
$$(j = 1,\, \ldots,\, n-1)$$
$$\Psi_j^{n,\,m-1} = \| \psi_{j1}^h,\, \ldots,\, \psi_{j,\,m-1}^h \| \quad (h = 0,\, l)$$

Systems (3.3) are solved successively. For $j = 1$ we have

$$A_2^{m-1} = M_2 A_1^{m-1} + \nu_2 a_{2m} + A_2^{F,\,m-1} \tag{3.4}$$

$$a_{1m} = \mu_2 A_1^{m-1} + \nu_{2m} a_{2m} + a_{1m}^F \tag{3.5}$$

where M_2 is a matrix of order $(m-1) \times (m-1)$, ν_2 and μ_2 are, respectively, column-and row-matrices of order $m-1$, and ν_{2m} is a number.

Relationships similar to (3.4) and (3.5) can evidently be obtained for any number j.

Thus, if the equalities

$$A_j^{m-1} = M_j A_1^{m-1} + \nu_j a_{jm} + A_j^{F,\,m-1} \tag{3.6}$$

$$a_{j-1,\,m} = \mu_j A_1^{m-1} + \nu_{jm} a_{jm} + a_{j-1}^F \tag{3.7}$$

hold, then by reducing the j-th system (3.3) to the form

$$\Psi_{j+1}^{0,\,m-1} A_{j+1}^{m-1} + (\psi_{jm}^l + \Psi_j^{l,\,m-1} \nu_j)\, a_{jm} =$$
$$-\Psi_j^{l,\,m-1} M_j A_1^{m-1} - \overset{\circ}{\psi}_{j+1,\,m} a_{j+1,\,m} + F_j - \Psi_j^{l,\,m-1} A_j^{F,\,m-1}$$

we can calculate

$$A_{j+1}^{m-1} = M_{j+1} A_1^{m-1} + \nu_{j+1} a_{j+1,\,m} + A_{j+1}^{F,\,m-1} \tag{3.8}$$

$$a_{jm} = \mu_{j+1} A_1^{m-1} + \nu_{j+1,\,m} a_{j+1,\,m} + a_{jm}^F \tag{3.9}$$

Solving Eqs. (3.9) in the reverse order, commencing with $j = n-1$, we obtain

$$a_{jm} = \mu^*_{j+1} A_1^{m-1} + \nu^*_{j+1,\,m} a_{nm} + a^{F*}_{jm} \quad (j = n-1, \ldots, 1) \tag{3.10}$$
$$\mu^*_{j+1} = \mu_{j+1} + \nu_{j+1,\,m}\mu^*_{j+2}, \quad \nu^*_{j+1,\,m} = \nu_{j+1,\,m}\nu^*_{j+2,\,m}$$
$$a^{F*}_{jm} = a^F_{jm} + \nu_{j+1,\,m} a^{F*}_{j+1,\,m}$$

Substituting the expressions (3.10) into equalities (3.6), we obtain

$$A_j^{m-1} = \mathrm{M}_1^* A_1^{m-1} + \nu_j^* a_{nm} + A^{F,\,m-1}_{j*} \quad (j = 2, \ldots, n) \tag{3.11}$$
$$\mathrm{M}_j^* = \mathrm{M}_j + \nu_j \mu^*_{j+1}, \quad \nu_j^* = \nu_j \nu^*_{j+1,\,m}, \quad A^{F,\,m-1}_{j*} = A^{F,\,m-1}_j + \nu_j a^{F*}_{jm}$$

Combining formulas (3.10) and (3.11) we obtain equality (3.2) where evidently

$$K_j' = \begin{Vmatrix} \mathrm{M}_j^* & \nu_j^* \\ \mu^*_{j+1} & \nu^*_{j+1,\,m} \end{Vmatrix}, \quad A_j^F = \begin{Vmatrix} A^{F,\,m-1}_{j*} \\ a^{F*}_{jm} \end{Vmatrix} \tag{3.12}$$

Although the algorithm is more complicated when the boundary effect is present, the volume of calculations is not increased. On the contrary, since the quantities ν_j and ν_{jm} are proportional to $\exp(-\lambda_j l_j)$, it can be reduced in the calculation of high-frequency oscillations.

For a closed system formulas (3.1) and (3.2) make possible the construction of the resolving system. Formulas (3.1) are applied in a manner described in [2, 3] for the method of initial parameters. The use of formulas (3.2) is similarly simple.

Let the closed system consist of n sections. In the absence of perturbation we have

$$A_1 = K_1'\{A_1^{m-1}, a_{nm}\}, \quad A_n = K_n'\{A_1^{m-1}, a_{nm}\}$$

From the conditions of junction of the n-th and first sections we obtain the system of m equations

$$\Psi_n{}^l A_n + \Psi_1{}^\circ A_1 = (\Psi_n{}^l K_n' + \Psi_1{}^\circ K_1')\{A_1^{m-1}, a_{nm}\} = 0$$

which is to be used for determining the natural frequencies and form of oscillations.

As an example of branched systems we shall consider the sequence of calculation of the free oscillations of the plane bar system shown in Fig. 1. Assuming that the system performs longitudinal bending or torsional bending oscillations, the state of its k-th branch can be defined by six parameters $A^{(k)} = \{a_1^{(k)}, \ldots, a_6^{(k)}\}$ $(k = 1, \ldots, 7)$. If each of the branches consists of several sections, then in the absence of the boundary effect $A^{(k)} = A_1^{(k)}$ ($A_1^{(k)}$ is the vector of parameters of the first section of the k-th branch), while in the presence of that effect $A^{(k)} = \{A_1^{(k)5}, a_{n6}^{(k)}\} \equiv \{a_{11}^{(k)}, \ldots, a_{15}^{(k)}, a_{n6}^{(k)}\}$, where parameters $a_{15}^{(k)}$ and $a_{n6}^{(k)}$ define boundary effects at the first section beginning and at the end of the last section. It is possible to eliminate from each branch 1, 2, 4, 6, and 7 three parameters, using the boundary conditions.

For example, for taking into account the boundary effect we express parameters $a_2^{(k)}$, $a_4^{(k)}$ and $a_5^{(k)}$ in terms of $a_1^{(k)}$, $a_3^{(k)}$ and $a_6^{(k)}$ $(k = 1, 2, 4, 6, 7)$ and, using the nine conditions at joint A, represent the nine parameters $a_1^{(1)}$, $a_3^{(1)}$, $a_6^{(1)}$, $a_1^{(2)}$, $a_3^{(2)}$, $a_6^{(2)}$, $a_2^{(3)}$, $a_4^{(3)}$, and $a_5^{(3)}$ in terms of the three $a_1^{(3)}$, $a_3^{(3)}$, and $a_6^{(3)}$. In the same way, using the conditions at joint B we express parameters $a_1^{(3)}$, $a_3^{(3)}$,

$a_6^{(3)}$, $a_1^{(4)}$, $a_3^{(4)}$, $a_6^{(4)}$, $a_2^{(5)}$, $a_4^{(5)}$, and $a_5^{(5)}$ in terms of $a_1^{(5)}$, $a_3^{(5)}$ and $a_6^{(5)}$. The nine equations that follow from conditions at joint C contain nine parameters $a_1^{(k)}$, $a_3^{(k)}$ and $a_6^{(k)}$ $(k = 5, 6, 7)$. We thus obtain a system which can be used for determining in the usual way the frequencies and forms of principal oscillations.

4. In a number of cases such as, for example, that of bars of variable cross section when, owing to the considerable number of sections into which the system must be divided, the method described above becomes somewhat complicated. However it can be considerably simplified by combining it with the method of slowly varying parameters [10, 11] or with some other variant of the perturbation method. It is then possible to represent a bar in the form of a combination of a small number of components within which bar parameters vary, but the vector of state can be represented with adequate accuracy by the formula

$$Y_i(x_i) = H_i(x_i) A_i + Y_i^F(x_i), \quad A_i = \{a_{i1}, \ldots, a_{im}\} \tag{4.1}$$

where i is the ordinal number of the bar component, $x_i \in [0, l_i]$ is its longitudinal coordinate, A_i is the vector of coefficients that are constant along the considered component, Y_i^F is the vector which appears when the perturbation effect is present. Matrix H_i which depends on the oscillation frequency and properties of the considered bar component is formulated so that the functions that define the boundary layer at the extremities of that component appear separately in it. When passing from one component to another formula (4.1) is subjected to the same operations as described in Sect. 2 and 3 relative to formulas (2.1).

One of the methods of deriving formula (4.1) is demonstrated below on the example of a straight bar of variable corss section performing transverse oscillations.

Let us assume that a straight bar contains a component (we omit its ordinal number for brevity) within whose boundaries the parameters are piecewise-continuous functions of the longitudinal component x with fairly small relative variations. Taking into consideration that in calculations it is more convenient to use sums instead of integrals, we approximate the considered component of the bar by a stepped bar which consists of a fairly large number of sections of constant cross section. External loads are assumed to be a system of concentrated forces $f(x)$ applied at joints of sections. Formulas (2.1) and (2.2) are valid for all sections of the bar. The x-coordinate is assumed to be the general coordinate for the whole of the considered component, and section ordinal numbers j are omitted. We further assume

$$\Lambda_x = \int_0^x \lambda(\zeta)\, d\zeta, \quad \Lambda = \Lambda(x) = \Lambda_x + \varphi, \quad \lambda(x) = \left[\frac{\omega^2 m(x)}{c(x)}\right]^{1/4}$$

where m, c, and λ are step functions. Vector A is evidently also a step function whose variation along the bar is determined by conditions of joining at cross sections $x = x^*$ where bar parameters change jumpwise and where external forces are applied. These conditions of joining are of the form

$$Y^+ = Y^- + F^*, \quad F^* = \{0, 0, 0, f(x^*)\}$$
$$(z^\pm = z(x^\pm), \quad x^\pm = x^* \pm 0)$$

Taking into account formula (2.1) we obtain $\Psi^+A^+ = \Psi^-A^- + F^*$ or

$$\Delta A = -(\Psi^+)^{-1}\Delta\Psi A^- + (\Psi^+)^{-1}F^* \quad (\Delta z = z^+ - z^-) \tag{4.2}$$

Using formula (2.2) we represent equality (4.2) in the expanded form

$$\begin{aligned}
\Delta a &= -\tfrac{1}{2}(\beta+\gamma)a^- + \tfrac{1}{2}(\beta-\gamma)(a^-\cos 2\Lambda^* - \\
&\quad b^-\sin 2\Lambda^*) + \beta(P^- + Q^-)\sin\Lambda^* + \\
&\quad [\beta'(-P^- + Q^-) + f_*]\cos\Lambda^* \\
\Delta b &= -\tfrac{1}{2}(\beta+\gamma)b^- - \tfrac{1}{2}(\beta-\gamma)(a^-\sin 2\Lambda^* + \\
&\quad b^-\cos 2\Lambda^*) + \beta(P^- + Q^-)\cos\Lambda^* - [\beta'(-P^- + Q^-) + \\
&\quad f_*]\sin\Lambda^*, \\
\Delta p &= -\tfrac{1}{2}(\beta+\gamma)p^- + \tfrac{1}{2}[(\gamma-\beta)Q^- + \beta v_1^- - \beta' u_1^- + \\
&\quad f_*]e^{\Lambda_*} \\
\Delta q &= -\tfrac{1}{2}(\beta+\gamma)q^- + \tfrac{1}{2}[(\gamma-\beta)P^- + \beta v_1^- + \beta' u_1^- - f_*]e^{\Lambda_l - \Lambda_*}
\end{aligned} \tag{4.3}$$

$$\beta = \langle c\lambda^2\rangle, \quad \gamma = \langle c\lambda^3\rangle + \langle\lambda\rangle, \quad \beta' = \langle c\lambda^3\rangle - \langle\lambda\rangle$$

$$(\langle z\rangle = \Delta z / 2z^+)$$

$$f_* = f(x_*) / 2(c\lambda^3)^+, \quad \Lambda^* = \Lambda(x^*), \quad \Lambda_* = \Lambda_{x^*},$$

$$P = p\exp(-\Lambda_x)$$

$$Q = q\exp(\Lambda_x - \Lambda_l), \quad v_1 = a\sin\Lambda + b\cos\Lambda$$

$$u_1 = a\cos\Lambda - b\sin\Lambda$$

In these equations it is possible to eliminate the first terms of their right-hand sides, and thus to separate out the asymptotic parts of functions $a(x)$, $b(x)$, $P(x)$, and $Q(x)$ (i.e. their principal parts when $\omega \to \infty$). We set $y = \eta y_\eta$ ($y = a, b, p, P, \ldots$), where the step function $\eta(x)$ is determined by the difference equation

$$\Delta\eta = -\tfrac{1}{2}(\beta+\gamma)\eta^- \tag{4.4}$$

Taking into the relationship $\Delta(\rho\sigma) = \rho^+\Delta\sigma + \sigma^-\Delta\rho$, instead of Eqs. (4.3) we have

$$\begin{aligned}
\Delta a_\eta &= \chi[\tfrac{1}{2}(\beta-\gamma)(a_\eta^-\cos 2\Lambda^* - b_\eta^-\sin 2\Lambda^*) + \\
&\quad \beta(P_\eta^- + Q_\eta^-)\sin\Lambda^* + \beta'(-P_\eta^- + Q_\eta^-)\cos\Lambda^*] + \Delta a_\eta^f \\
\Delta b_\eta &= \chi[-\tfrac{1}{2}(\beta-\gamma)(a_\eta^-\sin 2\Lambda^* + b_\eta^-\cos 2\Lambda^*) + \\
&\quad \beta(P_\eta^- + Q_\eta^-)\cos\Lambda^* - \beta'(-P_\eta^- + Q_\eta^-)\sin\Lambda^*] + \Delta b_\eta^f \\
\Delta p_\eta &= \frac{\chi}{2}[(\gamma-\beta)Q_\eta^- + \beta v_{1\eta}^- - \beta' u_{1\eta}^-]e^{\Lambda_*} + \frac{f_*}{2\eta^+}e^{\Lambda_*} \\
\Delta q_\eta &= \frac{\chi}{2}[(\gamma-\beta)P_\eta^- + \beta v_{1\eta}^- + \beta' u_{1\eta}^-]e^{\Lambda_l - \Lambda_*} - \frac{f_*}{2\eta^+}e^{\Lambda_l - \Lambda_*}
\end{aligned} \tag{4.5}$$

$$\chi = \frac{\eta^-}{\eta^+}, \quad \Delta a_\eta^f = \frac{f_*}{\eta^+}\cos\Lambda^*, \quad \Delta b_\eta^f = -\frac{f_*}{\eta^+}\sin\Lambda^*$$

In conformity with the assumption made above the quantities β, γ_1 and β' which define the relative variations of the bar parameters, are small. It follows from Eqs. (4.5) that for free and resonance oscillations the relative increments of elements of vector A_η also small. Hence restricting ourselves to approximate results we set in the

right-hand sides of Eqs. (4.5) $c_\eta^- = \bar{c}_\eta \equiv c_\eta(0)$ $(c = a, b, p)$ and $q_\eta^- = \bar{q}_\eta \equiv q_\eta(l)$. Carrying out summation and substituting for ease of calculation functions P and Q for p and q, we obtain the approximate relationships

$$A_{e\eta}(x) = K(x)\bar{A}_\eta + A_{e\eta}^f(x) \tag{4.6}$$

$$\bar{A}_\eta = \{\bar{a}_\eta, \bar{b}_\eta, \bar{p}_\eta, \bar{q}_\eta\}, \quad A_{e\eta} = \{a_\eta, b_\eta, P_\eta, Q_\eta\}$$

$$A_{e\eta}^f = \left\{\sum_0^x \Delta a_\eta^f, \sum_0^x \Delta b_\eta^f, \sum_0^x \frac{f_*}{2\eta^+} e^{\Lambda_* - \Lambda_x}, -\sum_x^l \frac{f_*}{2\eta^+} e^{\Lambda_x - \Lambda_*}\right\}$$

$$K(x) = \|k_{ij}(x)\|; \quad i, j = 1, 2, 3, 4$$

$$k_{11} = 1 + \varepsilon, \quad k_{22} = 1 - \varepsilon, \quad \varepsilon = \sum_0^x \frac{\chi}{2}(\beta - \gamma)\cos 2\Lambda^*$$

$$k_{12} = k_{21} = \sum_0^x \frac{\chi}{2}(\gamma - \beta)\sin 2\Lambda^*, \quad k_{33} = e^{-\Lambda_x}, \quad k_{44} = e^{\Lambda_x - \Lambda_l}$$

$$k_{13} = \sum_0^x \chi e^{-\Lambda_*}(\beta \sin\Lambda^* - \beta' \cos\Lambda^*), \quad k_{14} = \sum_0^x \chi e^{\Lambda_* - \Lambda_l}(\beta \sin\Lambda^* + \beta' \cos\Lambda^*), \quad k_{23} = \sum_0^x \chi e^{-\Lambda_*}(\beta \cos\Lambda^* + \beta' \sin\Lambda^*)$$

$$k_{24} = \sum_0^x \chi e^{\Lambda_* - \Lambda_l}(\beta \cos\Lambda^* - \beta' \sin\Lambda^*)$$

$$k_{31} = \sum_0^x \frac{\chi}{2} e^{\Lambda_* - \Lambda_x}(\beta \sin\Lambda^* - \beta' \cos\Lambda^*)$$

$$k_{32} = \sum_0^x \frac{\chi}{2} e^{\Lambda_* - \Lambda_x}(\beta \cos\Lambda^* + \beta' \sin\Lambda^*)$$

$$k_{34} = \sum_0^x \frac{\chi}{2}(\gamma - \beta) e^{2\Lambda_* - \Lambda_x - \Lambda_l}$$

$$k_{41} = \sum_x^l \frac{\chi}{2} e^{\Lambda_x - \Lambda_*}(\beta \sin\Lambda^* + \beta' \cos\Lambda^*)$$

$$k_{42} = \sum_x^l \frac{\chi}{2} e^{\Lambda_x - \Lambda_*}(\beta \cos\Lambda^* - \beta' \sin\Lambda^*)$$

$$k_{43} = \sum_x^l \frac{\chi}{2}(\gamma - \beta) e^{\Lambda_x - 2\Lambda_*}$$

The sums from 0 to x and from x to l extend over all quantities under the sign of summation which that, respectively appear in the intervals $[0, x)$ and $(x, l]$.

We denote by Ψ_e the matrix which differs from matrix Ψ by the substitution of unity for $\exp(-\Lambda_x)$ and $\exp(\Lambda_x - \Lambda_l)$. We can then set

$$Y = \Psi A = \Psi\eta A_\eta = \eta\Psi_e A_{e\eta}$$

With due regard to (4.6) we obtain the required expression

$$Y = \eta\Psi_e(K\bar{A}_\eta + A^f_{e\eta}) = H\bar{A}_\eta + Y^f \qquad (4.7)$$

$$H = \eta\Psi_e K, \quad Y^f = \eta\Psi_e A^f_{e\eta}$$

Remark 1. Let us assume that the bar parameters are continuous functions of the longitudinal coordinate. Then increasing infinitely the number of sections of the approximating step bar, instead of Eq. (4.4) we obtain

$$\frac{d\eta}{\eta} = -\frac{d(c\lambda^2)}{4c\lambda^2} - \frac{d(c\lambda^3)}{4c\lambda^3} - \frac{d\lambda}{4\lambda} = -\frac{d(c\lambda^3)}{2c\lambda^3}$$

which implies that $\eta\sqrt{c\lambda^3} = \text{const}$ or that for $\eta(0) = 1$ (cf. [11])

$$\eta(x) = \left[\frac{c_0\lambda_0^3}{c(x)\lambda^3(x)}\right]^{1/2} = \left[\frac{c_0 m_0^3}{c(x)m^3(x)}\right]^{1/8}$$

This formula can be used in the case of fairly small relative increments of the step bar along each of its steps. On this assumption we have $\beta' = \beta$.

Remark 2. The accuracy of formula (4.7) used for calculating free and resonant oscillations can be estimated on the basis of results in [11]. It can be shown, using the reasoning of [10], that the accuracy of formula (4.7) increases with increasing principal oscillation frequency. Generally this does not take place if for the elements of matrix (2.2) either hyperbolic or Krylov's functions are selected.

Although examples of bar systems oscillation were used here for illustrating the obtained basic results, the latter can be applied also for solving other boundary value problems of mechanics (such as oscillation of plates and shells, bending of beams of elastic supports, problems of the theory of elastic stability, etc.).

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